

Chaos in the Quantum Double Well Oscillator: The Ehrenfest View Revisited

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Abstract

We treat the double well quantum oscillator from the standpoint of the Ehrenfest equation but in a manner different from Pattanayak and Schieve. We show that for short times there can be chaotic motion due to quantum fluctuations, but over sufficiently long times the behaviour is normal.

It is generally agreed that the full quantum dynamics does not exhibit chaos. For systems which exhibit chaotic dynamics in the classical limit, it was clearly established by Fishman, Grempel and Prange¹ that there exists a critical time $t \sim 0$ ($1/h^{1/r}$) beyond which the dynamics crosses over to the quantal behaviour. The exponent ‘ r ’ was found to be 6.039 for systems which showed period-doubling bifurcations in the classical limit and 3.04 for the disappearance of the final KAM trajectory in the standard map. It was conjectured in the late eighties that there could be systems where the classical dynamics is obviously regular but the semiquantal dynamics can be chaotic (for a more precise explanation of the term “semiquantal”, c.f. Pattanayak and Schieve²). In support of this conjecture, Pattanayak and Schieve² explored the semiquantal dynamics of the double well oscillator governed by the Hamiltonian

$$H = \frac{P^2}{2} - \frac{1}{2} x^2 + \frac{\lambda}{4} x^4$$

The classical dynamics of this oscillator is obviously regular, as explained in Landau-Lifshitz (Vol. 1), being a periodic trajectory centered about $x = \pm \frac{1}{\sqrt{\lambda}}$ for total energy E in the range $0 > E > -1/4$ and about $x = 0$ for $E > 0$. It was shown, based on an Ehrenfest equation approach that in the semiquantal limit the quantum fluctuations cause the dynamics of this oscillator to become chaotic with four repelling zones in the phase space. Now, the full quantum dynamics of this oscillator should be regular just as all other fully quantum dynamics. Hence, we believe that the dynamics of this oscillator should cross over from a chaotic dynamics to a regular dynamics as one goes from a semiquantal to a fully quantum limit. The crossover should be characterized by a time t_0 . Unlike the cases treated by Grempel et al¹, we believe that the time scale here should be exponentially big. By using the Ehrenfest formulation, we will take the use of it by Pattanayak and Schieve² from a different standpoint. The Ehrenfest dynamics of the centroid $\langle x \rangle$ of a wave packet is given by

$$\langle \ddot{x} \rangle = \langle x \rangle - \lambda \langle x^3 \rangle,$$

i.e.

$$\langle \ddot{x} \rangle - \langle x \rangle + \lambda \langle x^3 \rangle = \lambda [\langle x^3 \rangle - \langle x^3 \rangle] = Q \quad (1)$$

Had the centroid followed the classical trajectory, the equations of motion would have been given by Eq. (1) with $Q = 0$. Q represents the effect of quantum fluctuations and acts as a

drive for the motion of the centroid of the wave packet. The effect of quantum mechanics on the dynamics is through the term Q . For short times, this drive is sinusoidal in time and that is when the dynamics is like that of a double well oscillator (Duffing Oscillator) in the presence of an oscillating field. This dynamics is known to be chaotic. At long times the drive changes character due to quantum interference effects and that is when the chaos induced by quantum fluctuation should go away.

In order to make the above point in a clearer fashion, we work with an average energy that is close to the ground state of the double well. With this deep an energy, tunneling has a low probability and there are two scales to the problem; one in which a wave packet oscillates inside a well and another in which it can tunnel from one well to another. With this in mind, the class of wave packets that we will work with is

$$\psi(x, t) = N_1(t)e^{-(x-a_0-\epsilon(t))^2/2b_0^2} + N_2(t)e^{-(x+a_0+\epsilon(t))^2/2b_0^2}. \quad (2)$$

We choose a_0 and b_0 such that with $\epsilon = 0$ and $N_1 = \pm N_2$, $\psi(x, t)$ yields the ground state and the first excited state in the variational sense. With $b_0 \ll a_0$ and to the lowest order in b_0^{-1} and $e^{-(a_0/b_0)^2}$, we have

$$a_0^2 + \frac{3}{2} b_0^2 = \frac{1}{\lambda} + \frac{\hbar^2}{4mb_0^6\lambda} e^{-a_0^2/b_0^2}. \quad (3)$$

Normalization yields

$$(N_1^2 + N_2^2 + 2N_1N_2e^{-(a_0^2/b_0^2)})b_0\sqrt{\pi} = 1. \quad (4)$$

Writing down $\langle H \rangle$ for the above state, we find that the two states with $\epsilon = 0$ and $N_1 = \pm N_2$ are separated by $\Delta E = \frac{a_0^2\hbar^2}{2mb_0^4} e^{-(a_0^2/b_0^2)}$ which equals $\frac{1}{\sqrt{\lambda}} e^{-\sqrt{2m}/\lambda\hbar}$, when we use Eq. (3), keeping in mind that \hbar is small. The time of quantum tunneling from one well to another is expected to be $\sqrt{\lambda} e^{\sqrt{2m}/\lambda\hbar}$ and over this time scale we expect the chaotic dynamics to be smoothened out.

Turning to the dynamics, we now have, using Eq. (2), the result

$$\langle x \rangle = (a_0 + \epsilon) \frac{|N_1|^2 - |N_2|^2}{|N_1|^2 + |N_2|^2 + 2\text{Re}(N_1^* N_2) e^{-(a_0+\epsilon)^2/b_0^2}} \quad (5)$$

and a similar expression may be found for $\langle x^3 \rangle$. We incorporate phase terms in N_1 and N_2 into the overall phase term of ψ . Thus we can say without loss of generality that $\{N_1, N_2\} \in \mathbb{R}$.

In keeping with our approximations, $e^{-(a_0+\epsilon)^2/b_0^2}$ is small and Eq. (5) can be taken to be (see Eq.(4) as well)

$$\langle x \rangle = (a_0 + \epsilon)(N_1^2 - N_2^2), \quad (6a)$$

$$\langle x^3 \rangle = (a_0 + \epsilon) \left[(a_0 + \epsilon)^2 + \frac{3}{2}b_0^2 \right] (N_1^2 - N_2^2). \quad (6b)$$

The quantum fluctuation Q in Eq.(1) is given by

$$\begin{aligned} Q = \lambda[\langle x \rangle^3 - \langle x^3 \rangle] &= \lambda \left\{ (a_0 + \epsilon)^3 (N_1^2 - N_2^2) [(N_1^2 - N_2^2)^2 - 1] \right\} \\ &\quad - \frac{3}{2} \lambda (a_0 + \epsilon) b_0^2 (N_1^2 - N_2^2). \end{aligned} \quad (6g)$$

The point to note is that, at small time scales, Q is a fast oscillation (i.e. same scale as frequency oscillation of $Q = 0$ set by oscillation in ϵ) and can induce³ chaos in the Duffing Oscillator. If E averages to 0, we have $\langle x \rangle \approx (N_1^2 - N_2^2)a_0$, $\langle x^3 \rangle = (N_1^2 - N_2^2)a_0(a_0^2 + \frac{3}{2}b_0^2)$ and Eq.(1) reads

$$\frac{d^2}{dt^2}(N_1^2 - N_2^2) + \lambda a_0 \left(a_0^2 + \frac{3}{2}b_0^2 - \frac{1}{\lambda} \right) (N_1^2 - N_2^2) = 0,$$

i.e.

$$\frac{d^2}{dt^2}(N_1^2 - N_2^2) + \frac{a_0 \hbar^2}{4mb_0^6} e^{-a_0^2/b_0^2} (N_1^2 - N_2^2) = 0. \quad (6h)$$

It is clear from Eq.(8) that the time scale over which $N_1^2 - N_2^2$ oscillates is $\mathcal{O}(e^{a_0^2/b_0^2})$ which is very big. In this limit, the quantum fluctuation Q of Eq.(7) shows an oscillatory behaviour with a long time scale. The effect of this low frequency drive is negligible.

For shorter time scales, where N_1^2 is $\gg N_2^2$ and is almost constant, we have $\langle x \rangle = (a_0 + \epsilon)$ and $\langle x^3 \rangle = (a_0 + \epsilon)^3 + \frac{3}{2}b_0^2(a_0 + \epsilon)$. This makes the dynamics of ϵ to be given by

$$\ddot{\epsilon} + 2a_0^2\lambda\epsilon + 3a_0\lambda\epsilon^2 + \lambda\epsilon^3 = 0. \quad (6i)$$

Since $a_0^2 \approx \frac{1}{\lambda}$, we have, for small λ ,

$$\ddot{\epsilon} + 2\epsilon = 0. \quad (6j)$$

The time period for the oscillations of ϵ is $\mathcal{O}(1)$. The quantum fluctuation term Q is now given by $Q \simeq -\frac{3}{2}\lambda(a_0 + \epsilon)b_0^2$, which is a drive with periodicity matching that of the double well with $Q = 0$. This drive is capable of introducing chaos³ and we believe that this is the

phenomenon reported by Pattanayak and Schieve². This crossover in the drive generated by quantum fluctuation is what we wanted to describe. While it is generally believed that quantum dynamics should be nonchaotic, there is a short time regime where the quantum fluctuation^{2,4} can make the centroid of a wave packet follow a classical trajectory which is chaotic. For sufficiently long times, the drive changes character and we do not have a chaotic response. The time scale for this crossover is exponentially large. It should be noted that if we had started with $N_2 \simeq 0$, after a very long time, we would find the system in a situation where $N_1 \approx 0$ and the chaotic phenomenon would return. This is another way of describing the quantum noise induced chaotic oscillations reported recently⁵.

References

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